第 11 卷 无锡轻工业学院学报 Vol. 11
第 3 期 JOURNAL OF THE WUXI INSTITUTE OF LIGHT INDUSTRY 1992 No. 3

具有两个弹性壁的一类相关随机游动在第 n 步吸收的概率母函数

费荣昌

(基础课部)

摘要 讨论了具有两个弹性壁的一类相关随机游动,求出了在第 n 步吸收的概率母函数的表达式,使参考文献[5,6]中的相应结果成为特殊情形。

关键词 相关随机游动;弹性壁;吸收概率;母函数

0 前 言

自 Goldstein 在[1]中提出相关随机游动以来,许多学者对各类相关随机游动进行了研究[2-5],文[5]讨论了两端具有吸收壁的相关随机游动,本文进一步研究两端具有弹性壁的相关随机游动,求出了在第 n 步吸收的概率母函数的表达式,使[5,6]中的相应结果成为本文的特殊情形。

一质点在一维有限整点 $0,1,\cdots,k$ 上随机游动,$t=0$ 时位于 $i(i=1,2,\cdots,k-1)$,$t=1,2,\cdots$ 时,质点的游动规律如下:

$$P\{\text{质点向右移动 1 单位}\}=\begin{cases}\alpha,\text{前一步向右},\\ \beta,\text{前一步向左},\end{cases}$$

$$P\{\text{质点向左移动 1 单位}\}=\begin{cases}1-\alpha,\text{前一步向右},\\ 1-\beta,\text{前一步向左},\end{cases}$$

其中 $0<\alpha,\beta<1$,在点 0 具有弹性壁,即当质点位于 1 时,

$$P\{\text{质点由 1 向右移动 1 单位}\}=\begin{cases}\alpha,\text{前一步向右},\\ \beta,\text{前一步向左},\end{cases}$$

$$P\{\text{质点由 1 向左移动,并弹回到 1}\}=\begin{cases}\delta_1(1-\alpha),\text{前一步向右},\\ \delta_1(1-\beta),\text{前一步向左},\end{cases}$$

$$P\{\text{质点由 1 向左移动 1 单位,并停留在 0}\}=\begin{cases}(1-\delta_1)(1-\alpha),\text{前一步向右},\\ (1-\delta_1)(1-\beta),\text{前一步向左}。\end{cases}$$

在点 k 也具有弹性壁,即当质点位于 $k-1$ 时,

收稿日期:1991-08-02

$$P\{质点由 k-1 向左移动 1 单位\}=\begin{cases}1-\alpha, 前一步向右, \\ 1-\beta, 前一步向左,\end{cases}$$

$$P\{质点由 k-1 向右移动, 并弹回到 k-1\}=\begin{cases}\delta_2\alpha, 前一步向右, \\ \delta_2\beta, 前一步向左,\end{cases}$$

$$P\{质点由 k-1 向右移动 1 单位, 并停留在 k\}=\begin{cases}(1-\delta_2)\alpha, 前一步向右, \\ (1-\delta_2)\beta, 前一步向左。\end{cases}$$

其中 $0\leqslant\delta_1,\delta_2<1$。称上述游动为两端具有弹性壁的相关随机游动。

下面来求这类随机游动在第 n 步吸收的概率母函数。

1 在第 n 步吸收的概率母函数

设 V_{in}, W_{in} 分别表示质点从 i 出发，到达 i 的前一步向右、向左，而在第 n 步被点 0 吸收的概率，则有

$$\begin{aligned} V_{i,n+1} &= \alpha V_{i+1,n} + (1-\alpha)W_{i-1,n}, \\ W_{i,n+1} &= \beta V_{i+1,n} + (1-\beta)W_{i-1,n}, \end{aligned} \qquad (i=1,2,\cdots,K-1) \tag{1}$$

边界条件为

$$\begin{aligned} W_{00} &= 1-\delta_1, \quad W_{0n} = \delta_1 V_{1n}, \\ V_{k0} &= 0, \quad V_{kn} = \delta_2 W_{k-1,n}. \end{aligned} \qquad (n=1,2,\cdots)$$

设 $V_i(s), W_i(S)$ 分别表示 V_{in}, W_{in} 的母函数，在(1)式两边乘以 S^{n+1}，再对 n 从 0 到∞求和，得

$$\begin{aligned} V_i(s) &= \alpha S V_{i+1}(s) + (1-\alpha)S W_{i-1}(s), \\ W_i(s) &= \beta S V_{i+1}(s) + (1-\beta)S W_{i-1}(s), \end{aligned} \tag{2}$$

边界条件为

$$W_0(s) - \delta_1 V_1(s) = 1-\delta_1, \quad V_k(s) - \delta_2 W_{k-1}(s) = 0.$$

当 $\alpha\neq\beta$ 时，(2)式化为

$$\begin{aligned} V_{i+1}(s) &= \frac{1-\beta}{(\alpha-\beta)S}V_i(s) - \frac{1-\alpha}{(\alpha-\beta)S}W_i(s), \\ W_{i+1}(s) &= \frac{\beta(1-\beta)}{\alpha(\alpha-\beta)S}V_i(s) + \frac{(\alpha-\beta)^2S^2-(1-\alpha)\beta}{\alpha(\alpha-\beta)S}W_i(s). \end{aligned} \tag{3}$$

以 $V_i(s) = A(s)\lambda^i(s), W_i(s) = B(s)\lambda^i(s)$ 代入(3)式得

$$\left[\lambda(s) - \frac{1-\beta}{(\alpha-\beta)S}\right]A(s) + \frac{1-\alpha}{(\alpha-\beta)S}B(s) = 0,$$

$$\frac{\beta(1-\beta)}{\alpha(\alpha-\beta)S}A(s) + \left[\frac{(\alpha-\beta)^2S^2-(1-\alpha)\beta}{\alpha(\alpha-\beta)S} - \lambda(s)\right]B(s) = 0. \tag{4}$$

方程组(4)有非零解的充要条件为

$$\lambda^2(s) - \frac{1+(\alpha-\beta)S^2}{\alpha S}\lambda(s) + \frac{1-\beta}{\alpha} = 0,$$

它的根为

$$\begin{matrix}\lambda_1(s)\\ \lambda_2(s)\end{matrix} = \frac{1}{2\alpha S}\{1+(\alpha-\beta)S^2\pm[(1+(\alpha-\beta)S^2)^2-4\alpha(1-\beta)S^2]^{1/2}\},$$

有

$$\lambda_1(s)\lambda_2(s) = \frac{1-\beta}{\alpha}.$$

设 $$V_i(s)=A_1(s)\lambda_1^i(s)+A_2(s)\lambda_2^i(s),$$

则由(3)
$$W_i(s)=\frac{1-\beta}{1-\alpha}\left[A_1(s)\lambda_1^i(s)+A_2(s)\lambda_2^i(s)\right]-\frac{(\alpha-\beta)S}{1-\alpha}\left[A_1(s)\lambda_1^{i+1}(s)+A_2(s)\lambda_2^{i+1}(s)\right]. \tag{5}$$

当 $\alpha=\beta$ 时,(5)式也成立。

由边界条件得

$$\left[\frac{1-\beta}{1-\alpha}-\left(\frac{\alpha-\beta}{1-\alpha}S+\delta_1\right)\lambda_1(s)\right]A_1(s)+\left[\frac{1-\beta}{1-\alpha}-\left(\frac{\alpha-\beta}{1-\alpha}S+\delta_1\right)\lambda_2(s)\right]A_2(s)=1-\delta_1,$$

$$\left[\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)\lambda_1(s)-\delta_2\frac{1-\beta}{1-\alpha}\right]\lambda_1^{k-1}(s)A_1(s)+\left[\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)\lambda_2(s)-\delta_2\frac{1-\beta}{1-\alpha}\right]\lambda_2^{k-1}(s)A_2(s)=0,$$

解得

$$A_1(s)=-(1-\delta_1)\left[\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)\lambda_2(s)-\delta_2\frac{1-\beta}{1-\alpha}\right]\lambda_2^{k-1}(s)/D(s),$$

$$A_2(s)=(1-\delta_1)\left[\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)\lambda_1(s)-\delta_2\frac{1-\beta}{1-\alpha}\right]\lambda_1^{k-1}(s)/D(s),$$

其中

$$\begin{aligned}D(s)=&\frac{1-\beta}{1-\alpha}\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)\left[\lambda_1^k(s)-\lambda_2^k(s)\right]\\&-\left\{\left[\delta_1+(1+\delta_1\delta_2)\frac{\alpha-\beta}{1-\alpha}S+\delta_2\left(\frac{\alpha-\beta}{1-\alpha}\right)^2S^2\right]\frac{1-\beta}{\alpha}\right.\\&\left.+\delta_2\left(\frac{1-\beta}{1-\alpha}\right)^2\right\}\left[\lambda_1^{k-1}(s)-\lambda_2^{k-1}(s)\right]\\&+\delta_2\frac{(1-\beta)^2}{\alpha(1-\alpha)}\left(\delta_1+\frac{\alpha-\beta}{1-\alpha}S\right)\left[\lambda_1^{k-2}(s)-\lambda_2^{k-2}(s)\right].\end{aligned}$$

于是

$$\begin{aligned}V_i(s)=&(1-\delta_1)\left(\frac{1-\beta}{\alpha}\right)^i\left\{\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)\left[\lambda_1^{k-i}(s)-\lambda_2^{k-i}(s)\right]\right.\\&\left.-\delta_2\frac{1-\beta}{1-\alpha}\left[\lambda_1^{k-i-1}(s)-\lambda_2^{k-i-1}(s)\right]\right\}/D(s),\end{aligned}$$

$$W_i(s)=\frac{1-\beta}{1-\alpha}V_i(s)-\frac{\alpha-\beta}{1-\alpha}SV_{i+1}(s).$$

设 g_{in} 表示质点从 i 出发,在第 n 步被点 0 吸收的概率,C_1,C_2 分别表示位于 i 时的前一步向右,向左的概率($C_1+C_2=1$),则

$$g_{in}=C_1V_{in}+C_2W_{in},$$

于是 g_{in} 的母函数为

$$\begin{aligned}G_i(s)&=C_1V_i(s)+C_2W_i(s)\\&=\left(C_1+C_2\frac{1-\beta}{1-\alpha}\right)V_i(s)-C_2\frac{\alpha-\beta}{1-\alpha}SV_{i+1}(s)\end{aligned}$$

$$
\begin{aligned}
&=(1-\delta_1)\left(\frac{1-\beta}{\alpha}\right)^i\left\{\left(C_1+C_2\frac{1-\beta}{1-\alpha}\right)\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)[\lambda_1^{k-i}(s)-\lambda_2^{k-i}(s)]\right.\\
&-\left[\delta_2\left(C_1+C_2\frac{1-\beta}{1-\alpha}\right)\frac{1-\beta}{1-\alpha}+C_2\frac{(1-\beta)(\alpha-\beta)}{\alpha(1-\alpha)}S\left(1+\delta_2\frac{\alpha-\beta}{1-\alpha}S\right)\right]\\
&[\lambda_1^{k-i-1}(s)-\lambda_2^{k-i-1}(s)]\\
&\left.+\delta_2C_2\frac{\alpha-\beta}{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^2S[\lambda_1^{k-i-2}(s)-\lambda_2^{k-i-2}(s)]\right\}/D(s).
\end{aligned}
\tag{6}
$$

在(6)式中分别以 $\delta_1,\delta_2,C_1,C_2,\alpha,1-\alpha,\beta,1-\beta,i,k-i$ 代替 $\delta_2,\delta_1,C_2,C_1,,1-\beta,\beta,1-\alpha,\alpha,k-i,i$，可得质点从 i 出发，在第 n 步被点 k 吸收的概率母函数为

$$
\begin{aligned}
H_i(s)&=(1-\delta_2)\left(\frac{\alpha}{1-\beta}\right)^{k-i}\left\{\left(C_1\frac{\alpha}{\beta}+C_2\right)\left(1+\delta_1\frac{\alpha-\beta}{\beta}S\right)[\lambda_1^{i}(s)-\lambda_2^{i}(s)]\right.\\
&-\left[\delta_1\left(C_1\frac{\alpha}{\beta}+C_2\right)\frac{\alpha}{\beta}+C_1\frac{\alpha(\alpha-\beta)}{(1-\beta)\beta}S\left(1+\delta_1\frac{\alpha-\beta}{\beta}S\right)\right][\lambda_1^{i-1}(s)-\lambda_2^{i-1}(s)]\\
&\left.+\delta_1C_1\frac{\alpha-\beta}{1-\beta}\left(\frac{\alpha}{\beta}\right)^2S[\lambda_1^{i-2}(s)-\lambda_2^{i-2}(s)]\right\}/E(s),
\end{aligned}
\tag{7}
$$

其中

$$
\begin{aligned}
E(s)&=\frac{\alpha}{\beta}\left(1+\delta_1\frac{\alpha-\beta}{\beta}S\right)[\lambda_1^{k}(s)-\lambda_2^{k}(s)]\\
&-\left\{\left[\delta_2+(1+\delta_1\delta_2)\frac{\alpha-\beta}{\beta}S+\delta_1\left(\frac{\alpha-\beta}{\beta}\right)^2S^2\right]\frac{\alpha}{1-\beta}+\delta_1\left(\frac{\alpha}{\beta}\right)^2\right\}\\
&[\lambda_1^{K-1}(s)-\lambda_2^{K-1}(s)]\\
&+\delta_1\frac{\alpha^2}{(1-\beta)\beta}\left(\delta_2+\frac{\alpha-\beta}{\beta}S\right)[\lambda_1^{k-2}(s)-\lambda_2^{k-2}(s)].
\end{aligned}
$$

2 特 例

当 $\delta_1=\delta_2=0$ 时，(6)、(7)即

$$
\begin{aligned}
G_i(s)&=\left(\frac{1-\beta}{\alpha}\right)^i\left\{[C_1(1-\alpha)+C_2(1-\beta)][\lambda_1^{k-i}(s)-\lambda_2^{k-i}(s)]\right.\\
&\left.-C_2\frac{(1-\beta)(\alpha-\beta)}{\alpha}S[\lambda_1^{k-i-1}(s)-\lambda_2^{k-i-1}(s)]\right\}/\\
&\left\{(1-\beta)[\lambda_1^{k}(s)-\lambda_2^{k}(s)]-\frac{(1-\beta)(\alpha-\beta)}{\alpha}S[\lambda_1^{k-1}(s)-\lambda_2^{k-1}(s)]\right\},
\end{aligned}
$$

$$
\begin{aligned}
H_i(s)&=\left(\frac{\alpha}{1-\beta}\right)^{k-i}\left\{(C_1\alpha+C_2\beta)[\lambda_1^{i}(s)-\lambda_2^{i}(s)]\right.\\
&\left.-C_1\frac{\alpha(\alpha-\beta)}{1-\beta}S[\lambda_1^{i-1}(s)-\lambda_2^{i-1}(s)]\right\}/\left\{\alpha[\lambda_1^{k}(s)-\lambda_2^{k}(s)]\right.\\
&\left.-\frac{\alpha(\alpha-\beta)}{1-\beta}S[\lambda_1^{k-1}(s)-\lambda_2^{k-1}(s)]\right\},
\end{aligned}
$$

这是[5]中两端具有吸收壁的相关随机游动的相应结果。

当 $C_1=\alpha=\beta=p,C_2=1-\alpha=1-\beta=q$ 时，(6)、(7) 即

$$G_i(s)=(1-\delta_1)\left(\frac{q}{p}\right)^i\{\lambda_1^{k-i}(s)-\lambda_2^{k-i}(s)-\delta_2[\lambda_1^{k-i-1}(s)-\lambda_2^{k-i-1}(s)]\}/$$

$$\left\{\lambda_1^k(s)-\lambda_2^k(s)-\left(\delta_1\frac{q}{p}+\delta_2\right)[\lambda_1^{k-1}(s)-\lambda_2^{k-1}(s)]\right.$$

$$\left.+\delta_1\delta_2\frac{q}{p}[\lambda_1^{k-2}(s)-\lambda_2^{k-2}(s)]\right\},$$

$$H_i(s)=(1-\delta_2)\left(\frac{p}{q}\right)^{k-i}\{\lambda_1^i(s)-\lambda_2^i(s)-\delta_1[\lambda_1^{i-1}(s)-\lambda_2^{i-1}(s)]\}/$$

$$\left\{\lambda_1^k(s)-\lambda_2^k(s)-\left(\delta_1+\delta_2\frac{p}{q}\right)[\lambda_1^{k-1}(s)-\lambda_2^{k-1}(s)]\right.$$

$$\left.+\delta_1\delta_2\frac{p}{q}[\lambda_1^{k-2}(s)-\lambda_2^{k-2}(s)]\right\},$$

这是两端具有弹性壁的经典随机游动的相应结果。

当 $\delta_1=\delta_2=0, C_1=\alpha=\beta=p, C_2=1-\alpha=1-\beta=q$ 时，(6)、(7) 即

$$G_i(s)=\left(\frac{q}{p}\right)^i\frac{\lambda_1^{k-i}(s)-\lambda_2^{k-i}(s)}{\lambda_1^k(s)-\lambda_2^k(s)},$$

$$H_i(s)=\left(\frac{p}{q}\right)^{k-i}\frac{\lambda_1^i(s)-\lambda_2^i(s)}{\lambda_1^k(s)-\lambda_2^k(s)},$$

这是[6]中两端具有吸收壁的经典随机游动的相应结果。

参 考 文 献

1 Goldstein S. On diffusion by discontinuons movements, and on telegraph equations. Quart J Mech, 1951;4:129—156

2 Renshaw E, Henderson R. The correlated random walk. J Appl Prob, 1981;18:403—414

3 Proudfoot A D, Lampard D G. A random walk problem with correlation. J Appl Prob, 1972; 9: 436—440

4 朱作宾. 一类相关随机游动. 应用概率统计, 1986;4:296—300

5 张元林. 具有两个吸收壁的一类相关随机游动在第 n 步吸收的概率母函数. 东南大学学报, 1989;4:121—125

6 费勒. 概率论及其应用(下册). 科学出版社, 1979

The Absorbing Probability Generating Functions at the nth Step for a Class of Correlated Random Walks with Two Elastic Barriers

Fei Rongchang

(Dept. of Fund. Course)

Abstract In this paper, we discuss a class of correlated random walks with two elastic barriers. For this class of walks we derive some expressions of the absorbing probability generating functions at the nth step. These results make the corresponding ones in [5, 6] a special case.

Key-words Correlated random walk; Elastic barrier; Absorbing probability; Generating function