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一类两个自由度系统的同宿轨道

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摘　要：研究了一类两个自由度系统，利用多重尺度法证明存在锁频于 Ω 的周期解，在一定条件下可变换为 Wiggins 的系统[1]，给出了这类系统的同宿轨道的计算公式.

关键词：两个自由度系统；同宿轨道；混沌；多重尺度法.

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Homoclinic Orbits of Some Systems Having Two Degrees of Freedom

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Abstract: Some systems having two degrees of freedom are investigated, the existence of the periodue solution locked to Ω is proved by using of the method of multiple scale. The systems can be translated to the systems developed by Wiggins[1] under some conditions. This calculating fommula for detecting the homoclinic orbits of this systems is given.

Key words: systems having two degrees; homoclinic orbits; chaos; method of multipe scale.

带有立方非线性的两个自由度系统在物理力学中应用很广，例如伸长度比较大的弦、梁膜和板的振动，动力隔振系统，动力消振器、球摆、向心摆和双摆的运动以及非线性弹簧相连的质量的运动等. 这些问题都可归结为带立方非线性两个自由度系统. 文献[2]主要用奇摄动方法讨论这类系统的周期解，还有人研究了一类特殊的两个自由度的 Hamilton 系统的混沌运动，但是这类问题的研究主要用数值方法.

Wiggins 等研究了一个四维常微分方程组，利用 Melnikov 方法给出了同宿轨道和多重脉动跳跃轨道的判别方法，并用于 Sine-Grodon 方程的两模截断形式中. 但他们考虑的 Bishop 截断形式[3]，所取系数相当特殊，既缺乏一般的讨论，也与实际的 Sine-Gordon 方程的数值结果相差甚远. 在广义渐近惯性流形[4]基础上得到了 Sine-Gordon 方程的约化形式，这是一个两个自由度带有立方非线性的系统，然后转化为复的常数微分方程组，再用类角动量变换转变为文献[1]中表述的形式.

作者利用文献[5]的思想研究了一类两个自由度系统的同宿轨道，得到了具体的判别同宿轨道的解析式，从而说明了这类系统的混沌机制.

考虑以下一类两个自由度系统：

$$\begin{cases} \ddot{u}_1 + w_1^2 u_1 = -2\hat{u}_1\dot{u}_1 + \alpha_1 u_1^3 + \alpha_3 u_1 u_2^2 + \\ \qquad \varepsilon\alpha_2 u_1^2 u_2 + \varepsilon\alpha_4 u_2^3 + F_1 cos(\Omega t + \tau_1), \\ \ddot{u}_2 + w_2^2 u_2 = -2\hat{u}_2\dot{u}_2 + \alpha_6 u_1^2 u_2 + \alpha_8 u_2^3 + \\ \qquad \varepsilon\alpha_5 u_1^3 + \varepsilon\alpha_7 u_1 u_2^2, \end{cases} \tag{1}$$

其中 $\Omega = w_1 + \varepsilon^2\sigma_1 = w_2 + \varepsilon^2\sigma_2$，利用多重尺度法得到锁频于 Ω 的周期解. 在第 2 节中令 $u_1 =$

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$\varepsilon B_1e^{i\Omega T}+\mathrm{cc.}$，$u_2=\varepsilon B_2e^{i\Omega T}+\mathrm{cc.}$，其中 cc. 表示复共轭项. 然后用类作用—角动量变换化成文献[1]讨论的四维常微分方程组. 在第 3 节中分析了未摄动方程的不变流形结构，并在第 4 节中研究了共振区附近的动力学行为，最后给出判断同宿轨道的解析式.

1　锁频于 Ω 的周期解

作者用多重尺度法研究(1)的周期解。令

$$u_1=\varepsilon u_{11}(T_0,T_2)+\varepsilon^3u_{13}(T_0,T_2)+\cdots,\tag{2}$$

$$u_2=\varepsilon u_{21}(T_0,T_2)+\varepsilon^3u_{23}(T_0,T_2)+\cdots,\tag{3}$$

其中 $T_n=\varepsilon^nt$，$\hat{\mu}_n=\varepsilon^2\mu_n$，$F_n=\varepsilon^3f_n$，$D_n=\frac{\partial}{\partial T_n}$，$n=1,2$. 代入(1)比较 ε 同次幂系数得解可表为

$$\begin{cases}u_{11}=A_1(T_2)\exp(iw_1T_0)+\mathrm{cc.},&(4)\\ u_{21}=A_2(T_2)\exp(iw_2T_0)+\mathrm{cc.},&(5)\end{cases}$$

$$\begin{cases}-2iw_1(A'_1+\mu_1A_1)+3\alpha_1A_1^2\overline{A}_1+2\alpha_3A_2\overline{A}_2A_1+\\ \alpha_3\overline{A}_1A_2^2\exp[2(\sigma_1-\sigma_2)T_2i]+\\ \frac{1}{2}f_1\exp[i(\sigma_1T_2+\tau_1)]=0, & (6)\\ -2iw_2(A'_2+\mu_2A_2)+3\alpha_8A_2^2\overline{A}_2+2\alpha_6A_1\overline{A}_1A_2+\\ \alpha_6\overline{A}_2A_1^2\exp[2(\sigma_2-\sigma_1)T_2i]=0 & (7)\end{cases}$$

令 $A_m=\frac{1}{2}a_m\exp(i\theta_m)$，$m=1,2$，其中 θ_m 和 a_m 为 T_2 的实函数，代入(6)，(7)得

$$\begin{cases}-2iw_1(\frac{1}{2}a'_1e^{i\theta_1}+\frac{1}{2}a_1e^{i\theta_1'}e^{i\theta_1}+\frac{1}{2}\mu_1a_1e^{i\theta_1})+\\ 3\alpha_1\frac{1}{8}a_1^3e^{i\theta_1}+2\alpha_3\frac{1}{8}a_2^2a_1e^{i\theta_1}+\\ \alpha_3\frac{a_1a_2^2}{8}e^{2i\theta_2-i\theta_1}e^{2i(\sigma_1-\sigma_2)T_2}+\\ \frac{1}{2}f_1\exp[i(\sigma_1T_2+\tau_1)]=0, & (8)\\ -2iw_2(\frac{1}{2}a'_2e^{i\theta_2}+\frac{1}{2}a_2i\theta_2'e^{i\theta_2}+\frac{1}{2}\mu_2a_2e^{i\theta_2})+\\ 3\alpha_8\frac{1}{8}a_2^3e^{i\theta_2}+2\alpha_6\frac{1}{8}a_1^2a_2e^{i\theta_2}+\\ \alpha_6\frac{1}{8}a_2a_1^2e^{2i\theta_1-i\theta_2}e^{2i(\sigma_2-\sigma_1)T_2}=0.\end{cases}$$

分离实虚部，并令 $r_1=2\theta_2-2\theta_1+2(\sigma_1-\sigma_2)T_2$，$r_2=\sigma_1T_2-\tau_1-\theta_1$，则(8)式转为

$$\begin{cases}-w_1(a'_1+\mu_1a_1)+\frac{\alpha_3}{8}a_1a_2^2\sin r_1+\\ \frac{1}{2}f_1\sin r_2=0,\\ r'_1=-\frac{3}{4w_2}\alpha_8a_2^2-\frac{1}{2w_2}a_1^2-\frac{\alpha_6}{4w_2}a_2a_1^2\cos r_1+\\ \quad\frac{3}{4w_1}\alpha_1a_1^2+\frac{1}{2w_1}\alpha_3a_2^2+\frac{\alpha_3}{4w_1}a_2^2\cos r_1+\\ \quad\frac{f_1}{w_1a_1}\cos r_2+(\sigma_1-\sigma_2), & (9)\\ -w_2(a'_2+\mu_2a_2)+\frac{\alpha_6}{8}a_2a_1^2\sin(-r_1)=0,\\ r'_2=\sigma_1+[\frac{3}{8w_1}\alpha_1a_1^2+\frac{1}{4w_1}\alpha_3a_2^2+\\ \quad\frac{\alpha_3}{8w_1}a_2^2\cos r_1+\frac{1}{2w_1a_1}f_1\cos r_2]. & (9)\end{cases}$$

对于静态解 $a'_n=r'_n=0$，$n=1,2$，从(9)可以求出 a_1、a_2，r_1、r_2，从而得到锁频于 Ω 的周期解.

$$u_1=\frac{\varepsilon}{2}a_1e^{i(\sigma_1T_2-\tau_1-r_2)}e^{iw_1T_0}+O(\varepsilon^3)=$$
$$\frac{\varepsilon}{2}a_1e^{i(-\tau_1-r_2)}e^{i\Omega T}+O(\varepsilon^3)+\mathrm{cc.}$$

$$u_2=\frac{\varepsilon}{2}a_2e^{i[\frac{r_1}{2}-(\sigma_1-\sigma_2)T_2+\theta_1)}e^{iw_2T_0}+O(\varepsilon^3)=$$
$$\frac{\varepsilon}{2}a_2e^{i(\frac{r_1}{2}-\tau_1-r_2)}e^{i\Omega T}+O(\varepsilon^3)+\mathrm{cc.}$$

2　类角动量—作用量变换

令 $u_i=\varepsilon B_ie^{i\Omega T_0}+\mathrm{cc.}$，则(1)可变换成

$$\begin{cases}D_2B_1=-\frac{i}{2\Omega}[2B_1w_1\sigma_1-2\mu_1B_1(i\Omega)+\\ \quad 3\alpha_1B_1^2\overline{B}_1+\alpha_3(2B_1B_2\overline{B}_2+\overline{B}_1B_2^2)+\\ \quad \varepsilon\alpha_2(B_1^2\overline{B}_2+2B_1\overline{B}_1B_2)+3\varepsilon\alpha_4B_2^2\overline{B}_2+\\ \quad \frac{f_1}{2}e^{i\tau_1}], & (10)\\ D_2B_2=-\frac{i}{2\Omega}[2B_2w_2\sigma_2-2\mu_2B_2(i\Omega)+\\ \quad \alpha_6(B_1^2\overline{B}_2+2B_1\overline{B}_1B_2)+3\alpha_8B_2^2\overline{B}_2+\\ \quad 3\varepsilon\alpha_5B_1^2\overline{B}_1+\varepsilon\alpha_7(2B_1B_2\overline{B}_2+\overline{B}_1B_2^2)].\end{cases}$$

设 $\alpha_6=\alpha_3$，$\alpha_2=3\alpha_5$，$\alpha_7=3\alpha_4$，$\mu_1=\varepsilon\hat{\mu}_1$，$\mu_2=\varepsilon\hat{\mu}_2$，$f_1=\varepsilon\hat{f}_1$. 令 $\varepsilon=0$，(10)式未摄动方程的 Hamilton 函数为

$$H_0=\frac{1}{2\Omega}[\sigma_1w_1B_1\overline{B}_1+\sigma_2w_2B_2\overline{B}_2]+\frac{3\alpha_1}{8\Omega}B_1^2\overline{B}_1^2+\frac{\alpha_3\overline{B}_1^2B_2^2}{8\Omega}+\frac{2\alpha_3B_1\overline{B}_1B_2\overline{B}_2}{4\Omega}+\frac{\alpha_6}{8\Omega}B_1^2\overline{B}_2^2+\frac{3\alpha_8}{8\Omega}B_2^2\overline{B}_2^2.\tag{11}$$

作类角动量—作用量变换，令 $B_1=|B_1|e^{-ir}$，$B_2=(x+iy)e^{-ir}$，$I=\frac{1}{2}(|B_1|^2+|B_2|^2)$，则(11)式为

$$H_0=\frac{1}{2\Omega}[\sigma_1w_1(2I-x^2-y^2)+\sigma_2w_2(x^2+y^2)]+\frac{3\alpha_2}{8\Omega}(2I-x^2-y^2)^2+\frac{\alpha_3}{8\Omega}(2I-x^2-y^2)\times 2(x^2-y^2)+\frac{2\alpha_3}{4\Omega}(2I-x^2-y^2)(x^2+y^2)+\frac{3\alpha_8}{8\Omega}(x^2+y^2)^2.\tag{12}$$

对应于 $\varepsilon\neq0$，在新变量形式下可得

$$
\begin{cases}
\dot{x}=\dfrac{\partial H_0}{\partial y}+\varepsilon\dfrac{\partial H_1}{\partial y}-\dfrac{\varepsilon\alpha_2\cdot 3xy}{2\Omega}\sqrt{2I-x^2-y^2}-\\
\quad\dfrac{3\varepsilon\alpha_4 y}{2\Omega}\dfrac{(x^2+y^2)x}{\sqrt{2I-x^2-y^2}}+\dfrac{\varepsilon\alpha_7\cdot 2xy}{2\Omega}-\mu_2 x,\\
\dot{y}=-\dfrac{\partial H_0}{\partial x}-\varepsilon\dfrac{\partial H_1}{\partial x}+\dfrac{3\varepsilon\alpha_2 x^2}{2\Omega}\sqrt{2I-x^2-y^2}+\\
\quad\dfrac{3\varepsilon\alpha_4}{2\Omega}\dfrac{(x^2+y^2)x^2}{\sqrt{2I-x^2-y^2}}-\dfrac{3\varepsilon\alpha_5}{2\Omega}(2I-x^2-y^2)^{\frac{3}{2}}-\\
\quad\dfrac{3\alpha_7\varepsilon}{2\Omega}\sqrt{2I-x^2-y^2}\times x^2-\dfrac{\alpha_7\varepsilon}{2\Omega}\times\\
\quad\sqrt{2I-x^2-y^2}\times y^2-\mu_2 y,\\
\dot{I}=-\varepsilon\dfrac{\partial H_1}{\partial r}-\mu_1(2I-x^2-y^2)+\mu_2(x^2+y^2),\\
\dot{r}=\dfrac{\partial H_0}{\partial I}+\varepsilon\dfrac{\partial H_1}{\partial I}+\dfrac{\varepsilon\alpha_2 3x}{2\Omega}\sqrt{2I-x^2-y^2}+\\
\quad 3\varepsilon\alpha_4\dfrac{(x^2+y^2)x}{\sqrt{2I-x^2-y^2}}.
\end{cases}
\tag{13}
$$

其中 $H_1=\dfrac{1}{4\Omega}\hat{f}_1\sqrt{2I-x^2-y^2}\cos(\tau_1+r)$，$\mu_1=\varepsilon\hat{\mu}_1$，$\mu_2=\varepsilon\hat{\mu}_2$，$f_1=\varepsilon\hat{f}_1$.

3　系统(13)的分析

先分析 $\varepsilon=0$ 时$(13)_{\varepsilon=0}$的不变流形结构. 由于 $\dot{I}=0$，故 x 和 y 方程解耦，可独立讨论，

$$
\begin{cases}
\dot{x}=\left(\dfrac{w_2\sigma_2-w_1\sigma_1-3\alpha_1 I+\alpha_3 I}{\Omega}\right)y+\\
\quad\left(\dfrac{3\alpha_1-4\alpha_3+3\alpha_8}{2\Omega}\right)x^2y+\left(\dfrac{3\alpha_1-2\alpha_3+3\alpha_8}{2\Omega}\right)\\
\quad y^3=M_1y+M_2x^2y+M_3y^3,\\
\dot{y}=\left(\dfrac{w_1\sigma_1-w_2\sigma_2+3\alpha_1 I-3\alpha_3 I}{\Omega}\right)x+\\
\quad\left(\dfrac{-3\alpha_1+4\alpha_3-3\alpha_8}{2\Omega}\right)xy^2+\left(\dfrac{-3\alpha_1+6\alpha_3-3\alpha_8}{2\Omega}\right)\\
\quad x^3=N_1x+N_2xy^2+N_3x^3.
\end{cases}
\tag{14}
$$

(14)式的奇点为(0,0)，$\left(0,\pm\sqrt{\dfrac{-M_1}{M_3}}\right)$，$\left(\pm\sqrt{\dfrac{N_1}{N_3}},0\right)$. 其中 Jacobi 行列式在(0,0)点处为

$$
\boldsymbol{J}=\begin{pmatrix}0 & M_1\\ N_1 & 0\end{pmatrix}
\tag{15}
$$

其行列式 $|\lambda\boldsymbol{E}-\boldsymbol{J}|=\lambda^2-M_1N_1=0$，当 $M_1N_1>0$ 则

$$
\begin{cases}
w_2\sigma_2-w_1\sigma_1-3\alpha_1 I+\alpha_3 I>0,\\
\sigma_1w_1-\sigma_2w_2+3\alpha_1 I-3\alpha_3 I>0,
\end{cases}
$$

或者
$$
\begin{cases}
w_2\sigma_2-w_1\sigma_1-3\alpha_1 I+\alpha_3 I<0,\\
\sigma_1w_1-\sigma_2w_2+3\alpha_1 I-3\alpha_3 I<0,
\end{cases}
\tag{16}
$$

如设 $w_2\sigma_2-w_1\sigma_1>0$，$\sigma_1<0$，$\alpha_1>0$ 时，则得

$\alpha_3<0$，$\alpha_1>\alpha_3$

或者　$\alpha_3>0$，$\alpha_1>\alpha_3$. I 满足(16).　　(17)

此时(0,0)为鞍点，$\left(0,\pm\sqrt{\dfrac{-M_1}{M_3}}\right)$为中心.

注意到过(0,0)时 $H_0=\dfrac{1}{2\Omega}\sigma_1w_1+\dfrac{3\alpha}{2\Omega}I^2$，(12)关于 x 轴和 y 轴对称，因而对应于 $H_0=\dfrac{\sigma_1w_1}{2\Omega}+\dfrac{3\alpha_1}{2\Omega}I^2$ 还存在一对连结(0,0)的同宿轨道，于是可得(14)的相图，见图1.

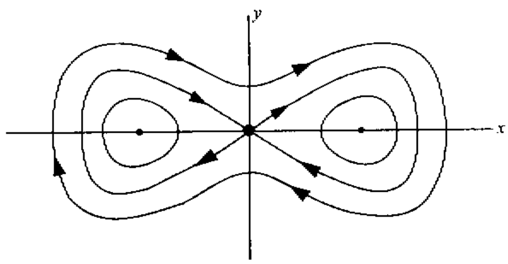

图1　同宿轨道

Fig.1　Homoclinic orbits

这样对于四维系统$(13)_{\varepsilon=0}$，存在正规双曲不变流形

$$\mu=\{(x,y,I,r)\mid x=y=0,(17)\text{成立}\}.\tag{18}$$

记 $w^s(\mu)$和 $w^u(\mu)$为 μ 的稳定流形和不稳定流形，则 μ 在四维空间中存在三维同宿轨道构成的不变集.

$w^s(\mu)\cap w^u(\mu)\equiv\{(x,y,I,r)\mid H_0(x,y,r)-H_0(0,0,I)=0,(17)\text{成立}\}$.

可以用如下方法求出 $w^s(\mu)\cap w^u(\mu)$同宿轨道. 令

$$x+iy=\sqrt{2B}e^{i\theta},\tag{19}$$

则(14)式转为

$$
\begin{cases}
\dot{B}=(M_1+N_1)B\sin2\theta+\dfrac{\alpha_3}{2\Omega}(2B)^2\sin2\theta,\\
\dot{\theta}=\left(\dfrac{N_1-M_1}{2}\right)-\dfrac{\alpha_3 I}{\Omega}\cos2\theta+\\
\quad B\left(2N_2+\dfrac{2\alpha_3}{\Omega}\cos2\theta\right);
\end{cases}
\tag{20}
$$

$$
\begin{aligned}
H_0=&\frac{1}{\Omega}[\sigma_1w_1(I-B)+\sigma_2w_2B]+\\
&\frac{3\alpha}{8\Omega}(2I-2B)^2+\frac{\alpha_3}{8\Omega}(I-B)4(2B\cos2\theta)+\\
&\frac{2\alpha_3}{4\Omega}(2I-2B)(2B)+\frac{3\alpha_8}{8\Omega}(2B)^2.
\end{aligned}
\tag{21}
$$

在同宿轨道上

$$
B=\frac{-\left(\dfrac{\sigma_2w_2-\sigma_1w_1}{\Omega}\right)+\dfrac{3\alpha_1 I}{\Omega}--\dfrac{\alpha_3 I}{\Omega}\cos2\theta-\dfrac{2\alpha_3 I}{\Omega}}{\dfrac{3\alpha_1}{2\Omega}-\dfrac{\alpha_3}{\Omega}\cos2\theta-\dfrac{2\alpha_3}{\Omega}+\dfrac{3\alpha_8}{2\Omega}}\triangleq
\frac{\theta_1+\theta_2\cos2\theta}{\theta_3+\theta_4\cos2\theta},
\tag{22}
$$

$$
\begin{aligned}
\dot{\theta} = {} & \left(\frac{N_1 - M_1}{2}\right) + \frac{2(\sigma_2 w_2 - \sigma_1 w_1)}{\Omega} + \\
& \frac{4\alpha_3 - 6\alpha_1}{\Omega} I + \frac{\alpha_3 I}{\Omega}\cos 2\theta = \\
& \frac{w_2\sigma_2 - w_1\sigma_1 - 3\alpha_1 I + 2\alpha_3 I}{\Omega} + \\
& \frac{\alpha_3 I}{\Omega}\cos 2\theta \triangleq P_1 + P_2\cos 2\theta
\end{aligned} \tag{23}
$$

即 $$\int \frac{\mathrm{d}\theta}{P_1 + P_2\cos 2\theta} = T_2 + c.$$

令 $\tan\theta = z$，

$$\int \frac{\frac{1}{1+z^2}\mathrm{d}z}{P_1 + P_2\left(\frac{1-z^2}{1+z^2}\right)} =$$

$$\int \frac{\mathrm{d}z}{(P_1+P_2)\left(1 - \frac{P_2 - P_1}{P_1 + P_2} z^2\right)} = T_2 + c.$$

当 $$P_1 + P_2 = \frac{w_2\sigma_2 - w_1\sigma_1 - 3\alpha_1 I + 3\alpha_3 I}{\Omega} > 0,$$

$$P_2 - P_1 = \frac{w_1\sigma_1 - w_2\sigma_2 + 3\alpha_1 I - \alpha_3 I}{\Omega} > 0,$$

在 $\left|\sqrt{\frac{P_2 - P_1}{P_1 + P_2}}\, z\right| < 1$ 条件下，

$$z = \sqrt{\frac{P_1 + P_2}{P_2 - P_1}}\,\mathrm{th}\sqrt{P_2^{\,2} - P_1^{\,2}}(T_2 + c); \tag{24}$$

在 $\left|\sqrt{\frac{P_2 - P_1}{P_1 + P_2}}\, z\right| > 1$ 条件下，

$$z = \sqrt{\frac{P_1 + P_2}{P_2 - P_1}}\,\mathrm{cth}\sqrt{P_2^{\,2} - P_1^{\,2}}(T_2 + c); \tag{25}$$

取初始条件 $c = 0$，当对于(24)时((25)情况类似)，

$$\tan^2\theta = \left(\frac{P_1 + P_2}{P_2 - P_1}\right)\left[\frac{\mathrm{ch}2\sqrt{P_2^{\,2} - P_1^{\,2}}\,T_2 - 1}{\mathrm{ch}2\sqrt{P_2^{\,2} - P_1^{\,2}}\,T_2 + 1}\right],$$

于是

$$
\begin{aligned}
B = {} & \Big((\theta_1 + \theta_2)(P_2 - P_1)(\mathrm{ch}2\sqrt{P_2^{\,2} - P_1^{\,2}}\,T_2 + 1) + \\
& (\theta_1 - \theta_2)(P_1 + P_2)(\mathrm{ch}2\sqrt{P_2^{\,2} - P_1^{\,2}}\,T_2 - 1)\Big) \Big/ \\
& \Big((\theta_3 + \theta_4)(P_2 - P_1)(\mathrm{ch}2\sqrt{P_2^{\,2} - P_1^{\,2}}\,T_2 + 1) + \\
& (\theta_3 - \theta_4)(P_1 + P_2)(\mathrm{ch}2\sqrt{P_2^{\,2} - P_1^{\,2}}\,T_2 - 1)\Big)
\end{aligned} \tag{26}
$$

当 $I = I_r = -\frac{\sigma_1 w_1}{3\alpha_1}$ 时，

$$\frac{\mathrm{d}r}{\mathrm{d}T_2} = \left(\frac{-3\alpha_1 + 2\alpha_3}{\Omega}\right)B + \frac{\alpha_3}{\Omega}B\cos 2\theta,$$

$$\triangle r = \int_{-\infty}^{+\infty}\left[\begin{array}{c}\left(\frac{-3\alpha_1 + 2\alpha_3}{\Omega}\right)B + \\ \frac{\alpha_3}{\Omega}B\cos 2\theta\end{array}\right]\mathrm{d}T_2, \tag{27}$$

利用(19)(24)(26)可求得同宿轨道.

当 $\varepsilon \neq 0$ 时，在 ε 充分小情况下可选取开集：

$$\bigcup^{\delta} = \{(x, y, I, r) \mid |x| < \delta, |y| < \delta, \tilde{I}_1 \leqslant I \leqslant \tilde{I}_2\},$$

其中共振值 $I_r = -\frac{\sigma_1 w_1}{3\alpha_1} \in [\tilde{I}_1, \tilde{I}_2]$，(13)式也存在正规双曲不变集

$$\mu_\varepsilon = \left\{\begin{array}{l}(x, y, I, r) \left| \begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon \begin{bmatrix} x_1(I, r) \\ y_1(I, r) \end{bmatrix} + \right. \\ 0(\varepsilon^2), \tilde{I}_1 \leqslant I \leqslant \tilde{I}_2. \end{array}\right\}$$

它有与 $w^s\mathrm{loc}(\mu)$ 和 $w^u\mathrm{loc}(\mu)$ 充分接近的局部稳定和局部不稳定流形.

4　共振区附近的动力学行为

在 $w^s(\mu) \cap w^u(\mu)$ 上关于 $\dot{r}$ 的方程为

$$\dot{r} = \frac{\sigma_1 w_1}{\Omega} + \frac{3\alpha_1 I}{\Omega}. \tag{28}$$

因而在 $I = I_r = -\frac{\sigma_1 w_1}{3\alpha_1}$ 处要产生共振. 在 $I = I_r = -\frac{\sigma_1 w_1}{3\alpha_1}$ 附近定义一个环域：

$$A_\varepsilon = \left\{\begin{array}{l}(x, y, h, r) \left| \begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon \begin{bmatrix} x_1(I_r + \sqrt{\varepsilon}h) \\ y_1(I_r + \sqrt{\varepsilon}h) \end{bmatrix} + \right. \\ 0(\varepsilon^2), |h| < c \end{array}\right\}. \tag{29}$$

其中 $c > 0$ 为常数. A_ε 的三维稳定和不稳定流形为 $w^s(A_\varepsilon)$ 和 $w^u(A_\varepsilon)$，它们为 $w^s(\mu_\varepsilon)$ 和 $w^u(\mu_\varepsilon)$ 的子集. 令 $\rho = \sqrt{\varepsilon}T_2$，$I = I_r + \sqrt{\varepsilon}h$，$A_\varepsilon$ 上的动力学由以下方程决定：

$$
\left\{
\begin{aligned}
h' = {} & -2\hat{\mu}_1 I_r + \frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1\sin(\tau_1 + r) + \\
& [-2\hat{\mu}_1\sqrt{\varepsilon} + \frac{1}{4\Omega}\hat{f}_1\sqrt{\varepsilon}h\sin(\tau_1 + r)], \\
r' = {} & \frac{3\alpha_1}{\Omega}h + \frac{1}{\sqrt{2I_r}}\frac{1}{4\Omega}\sqrt{\varepsilon}\hat{f}_1\cos(\tau_1 + r).
\end{aligned}
\right. \tag{30}
$$

去掉 $\sqrt{\varepsilon}$ 扰动项即得

$$
\left\{
\begin{aligned}
h' &= -2\hat{\mu}_1 I_r + \frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1\sin(\tau_1 + r), \\
r' &= \frac{3\alpha_1}{\Omega}h.
\end{aligned}
\right. \tag{31}
$$

(31)式的 Hamilton 函数为

$$
\begin{aligned}
\mathscr{H} = {} & \frac{3\alpha_1}{2\Omega}h^2 - \frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1\cos(\tau_1 + r) - \\
& 2\hat{\mu}_1 I_r r.
\end{aligned} \tag{32}
$$

(31)式的平衡点为

$$q_0\left[0, \arcsin\left[\frac{2\hat{\mu}_1 I_r}{\frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1}\right] - \tau_1\right],$$

$$p_0\left[0, \pi - \arcsin\frac{2\hat{\mu}_1 I_r}{\frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1} + \tau_1\right]. \tag{33}$$

相图为图 2，p_0 为中心，q_0 为鞍点；(30)式的相图为

图3,其中 q_ε,p_ε 为 q_0,p_0 小扰动的结果.

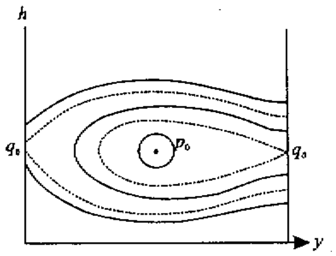

图2　无扰动项的轨道

Fig.2　No pertarbing orbits

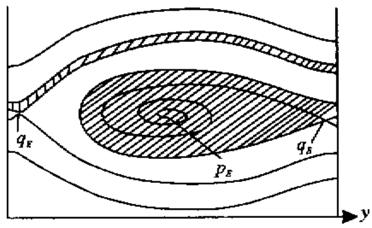

图3　带扰动的轨道

Fig.3　Perturbing orbits

5　同宿轨道存在性的判定

现在 Melnikov 函数为

$$M(I^r,\hat{\mu}_1,\hat{\mu}_2,\hat{f}_1)=\int_{-\infty}^{+\infty}\Big\{\frac{\partial H_0}{\partial x}\cdot\frac{\partial H_1}{\partial y}-\frac{\partial H_0}{\partial y}\cdot\frac{\partial H_1}{\partial x}+\left(\frac{\partial H_0}{\partial I}(x,y,I)-\frac{\partial H_0}{\partial I}(0,0,I)\right)\times\frac{\partial H_1}{\partial r}-\hat{\mu}_2\left(x\frac{\partial H_0}{\partial x}+y\frac{\partial H_1}{\partial y}\right)+\frac{\partial H_0}{\partial x}\times\Big[-\frac{3\alpha_2 xy}{2\Omega}\sqrt{2I-x^2-y^2}+\frac{3\alpha_4 xy(x^2+y^2)}{2\Omega\sqrt{2I-x^2-y^2}}+\frac{2\alpha_7 xy}{2\Omega}\Big]+\frac{\partial H_0}{\partial y}\Big[\frac{3\alpha_2 x^2}{2\Omega}\sqrt{2I-x^2-y^2}+\frac{3\alpha_4(x^2+y^2)x^2}{2\Omega\sqrt{2I-x^2-y^2}}-\frac{3\alpha_5}{2\Omega}(2I-x^2-y^2)^{3/2}-\frac{1}{2\Omega}\alpha_7(2I-x^2-y^2)(x^2+y^2)-\frac{\alpha_7}{2\Omega}\sqrt{2\Omega-x^2-y^2}\cdot y^2\Big]+\Big(\frac{\partial H_0}{\partial I}(x,y,I)-\frac{\partial H_0}{\partial I}(0,0,I)\Big)-\hat{\mu}_1(2I-x^2-y^2)-\hat{\mu}_2(x^2+y^2)\Big\}\mathrm{d}T_2. \tag{34}$$

考虑到 $\frac{\partial H_0}{\partial x}\cdot\frac{\partial H_1}{\partial y}-\frac{\partial H_0}{\partial y}\cdot\frac{\partial H_1}{\partial x}=-\frac{\mathrm{d}H_1}{\mathrm{d}t}-\frac{\partial H_1}{\partial r}\times\frac{\partial H_0}{\partial I}$,$I=I_r$ 时,$\frac{\partial H_0}{\partial I}(0,0,I)=0$ 以及函数的奇偶性,(34)式转为

$$\int_{-\infty}^{+\infty}\Big\{-\frac{\mathrm{d}H_1}{\mathrm{d}T_2}-\hat{\mu}_2(x\dot{y}-\dot{x}y)+\dot{r}\big[-2\hat{\mu}_1 I_r+(x^2+y^2)(\hat{\mu}_1-\hat{\mu}_2)\big]\mathrm{d}T_2=-H_1\Big|_{-\infty}^{+\infty}-2\hat{\mu}_1\triangle\gamma I_r+\hat{\mu}_2 I_0+(\hat{\mu}_1-\hat{\mu}_2)I_1=0, \tag{35}$$

其中

$$-H_1\Big|_{-\infty}^{+\infty}=-\frac{1}{4\Omega}\hat{f}_1\sqrt{2I_r}\big[\cos(\tau_1+r(-\infty))\big]\times(\cos\triangle r-1)-\sin(\tau_1+r(-\infty))\big]\sin\triangle r, \tag{36}$$

$$\cos(\tau_1+r(-\infty))=\pm\sqrt{1-\frac{3\Omega\hat{\mu}_1{}^2 I_r}{\hat{f}^2}},$$

$$\sin(\tau_1+r(-\infty))=\frac{8\Omega\hat{\mu}_1\sqrt{I_r}}{\sqrt{2}f_2},$$

$$I_0=\int_{-\infty}^{+\infty}(\dot{y}x-\dot{x}y)\mathrm{d}T_2=\int_{-\infty}^{+\infty}2B\mathrm{d}\theta=2\int_{-\infty}^{+\infty}\frac{\theta_1+\theta_2\cos2\theta}{\theta_3+\theta_4\cos2\theta}\mathrm{d}\theta,$$

$$I_1=\int_{-\infty}^{+\infty}(x^2+y^2)\dot{r}\mathrm{d}T_2=\int_{-\infty}^{+\infty}2B\mathrm{d}r.$$

联结 p_ε 的同宿轨道的存在性首先要求 Melnikov 函数(35)等于0,这可给出比值 $\frac{\hat{\mu}_1}{\hat{f}_1}$,$\frac{\hat{\mu}_2}{\hat{f}_1}$.然后要求从 p_ε 出发的轨道回到图1相图的'鱼'曲线之中,见图4.

图4中鱼头部 r_n,鱼尾部 r_s 必须满足

$$\mathscr{H}(0,r)-\mathscr{H}(0,r_s)=0,$$

即　$r_n<r^\infty<r_s$,

$$\mathscr{H}(0,r_n)-\mathscr{H}\Big(0,2\pi+\arcsin\frac{2\hat{\mu}_1 I_r}{\frac{1}{4\Omega}\sqrt{2I_r}f_1}-\tau_1\Big)=-\frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1\cos(\tau_1+r_n)+\frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1\times\cos\Big(2\pi+\arcsin\frac{2\hat{\mu}_1 I_r}{\frac{1}{4\Omega}\sqrt{2I_r}\hat{f}_1}-\tau_1\Big)=0.$$

从中可解出 r_n,选择参数可使

$$r_n<r_{p_0}+\triangle r<r_{q_0}. \tag{37}$$

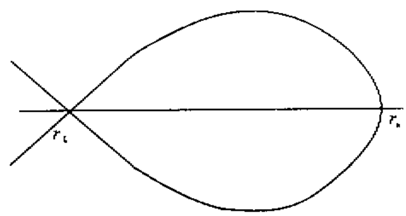

图4　同宿轨道

Fig.4　Homoclinic orbits

因此得到定理如下:

定理　当 ε 充分小,选择(1)参数使(37)式与(35)

式同时成立,则(1)式具有同宿轨道.

实例　取 $\Omega=1$, $w_1\sigma_1=-1$, $\mu_1=\varepsilon\alpha$, $\alpha_1=\frac{1}{6}$, $\alpha_3=\frac{1}{2}$, $f_1=4\varepsilon T$, $\tau_1=-\frac{\pi}{2}$, $w_2\sigma_2=-(1+K^2)$, $\mu_2=\varepsilon\beta$, $\alpha_6=\frac{1}{2}$, $\alpha_8=\frac{1}{4}$,则(1)式的变形方程(15)式为

$$iD_2B_1=\frac{1}{2}|B_1|^2B_1+\frac{1}{2}|B_2|^2B_1-B_1+\frac{1}{2}B_1|B_2|^2+\frac{1}{2}\overline{B}_1B_2{}^2-i\varepsilon\alpha B_1-i\varepsilon T+\varepsilon\alpha_2(B_1^2\,\overline{B}_2+2B_1\,\overline{B}_1B_2)+3\varepsilon\alpha_4B_2^2\,\overline{B}_2,$$

$$iD_2B_2=\frac{1}{2}|B_1|^2B_2+\frac{3}{4}|B_2|^2B-(1+K^2)B_1+\frac{1}{2}(B_1\,\overline{B}_2+\overline{B}_1B_2)B_1-i\varepsilon\beta B_2+3\varepsilon\alpha_5B_1^2\,\overline{B}_1+\varepsilon\alpha_7(2B_1B_2\,\overline{B}_2+\overline{B}_1B_2{}^2).$$

这类似于文献[1]研究过的(6.1),可利用定理计算出存在同宿轨道的参数条件.

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